

# On the intersection of solvable Hall subgroups in finite simple exceptional groups of Lie type\*

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## Abstract

Assume that a finite almost simple group with simple socle isomorphic to an exceptional group of Lie type possesses a solvable Hall subgroup. Then there exist four conjugates of the subgroup such that their intersection is trivial.

**Keywords:** almost simple group, base size, solvable Hall subgroup.

## Introduction

Throughout the paper the term “group” we always use in the meaning “finite group”. We use symbols  $A \leq G$  and  $A \trianglelefteq G$  if  $A$  is a subgroup of  $G$  and  $A$  is a normal subgroup of  $G$  respectively. If  $\Omega$  is a (finite) set, then by  $\text{Sym}(\Omega)$  we denote the group of all permutations of  $\Omega$ . We also denote  $\text{Sym}(\{1, \dots, n\})$  by  $\text{Sym}_n$ . Given  $H \leq G$  by  $H_G = \bigcap_{g \in G} H^g$  we denote the kernel of  $H$ .

Assume that  $G$  acts on  $\Omega$ . An element  $x \in \Omega$  is called a *G-regular point*, if  $|xG| = |G|$ , i.e., if the stabilizer of  $x$  is trivial. We define the action of  $G$  on  $\Omega^k$  by

$$g : (i_1, \dots, i_k) \mapsto (i_1g, \dots, i_kg).$$

If  $G$  acts faithfully and transitively on  $\Omega$ , then the minimal  $k$  such that  $\Omega^k$  possesses a  $G$ -regular point is called the *base size* of  $G$  and is denoted by  $\text{Base}(G)$ . For every natural  $m$  the number of  $G$ -regular orbits on  $\Omega^m$  is denoted by  $\text{Reg}(G, m)$  (this number equals 0 if  $m < \text{Base}(G)$ ). If  $H$  is a subgroup of  $G$  and  $G$  acts on the set  $\Omega$  of right cosets of  $H$  by right multiplications, then  $G/H_G$  acts faithfully and transitively on  $\Omega$ . In this case we denote  $\text{Base}(G/H_G)$  and  $\text{Reg}(G/H_G, m)$  by  $\text{Base}_H(G)$  and  $\text{Reg}_H(G, m)$  respectively. We also say that  $\text{Base}_H(G)$  is the *base size of  $G$  with respect to  $H$* . Clearly,  $\text{Base}_H(G)$  is the minimal  $k$  such that there exist elements  $x_1, \dots, x_k \in G$  with  $H^{x_1} \cap \dots \cap H^{x_k} = H_G$ . Thus, the base size of  $G$  with respect to  $H$  is the minimal  $k$  such that there exist  $k$  conjugates of  $H$  with intersection equals  $H_G$ .

We prove the following theorem in the paper.

**Theorem 1.** (Main Theorem) *Let  $G$  be an almost simple group with simple socle isomorphic to an exceptional group of Lie type. Assume also that  $G$  possesses a solvable Hall subgroup  $H$ . Then  $\text{Base}_H(G) \leq 4$ .*

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\*The work is supported by RFBR, projects 11–01–00456, 12–01–33102.

The following results were obtained in this direction. In 1966 D.S.Passman proved (see [1]) that a  $p$ -solvable group possesses three Sylow  $p$ -subgroups whose intersection equals the  $p$ -radical of  $G$ . Later in 1996 V.I.Zenkov proved (see [2]) that the same conclusion holds for arbitrary finite group  $G$ . In [3] S.Dolfi proved that in every  $\pi$ -solvable group  $G$  there exist three conjugate  $\pi$ -Hall subgroups whose intersection equals  $O_\pi(G)$  (see also [4]). Notice also that V.I.Zenkov in [5] constructed an example of a group  $G$  possessing a solvable  $\pi$ -Hall subgroup  $H$  such that the intersection of five conjugates of  $H$  equals  $O_\pi(G)$ , while the intersection of every four conjugates of  $H$  is greater than  $O_\pi(G)$ . In [6, Theorem 1] the following statement is proven.

**Theorem 2.** *Let  $G$  be a finite group possessing a solvable  $\pi$ -Hall subgroup  $H$ . Assume that for every simple component  $S$  of  $E(\overline{G})$  of the factor group  $\overline{G} = G/S(G)$ , where  $S(G)$  is a solvable radical of  $G$ , the following condition holds:*

*for every  $L$  such that  $S \leq L \leq \text{Aut}(S)$  and contains a solvable  $\pi$ -Hall subgroup  $M$ ,  
the inequalities  $\text{Base}_M(L) \leq 5$  and  $\text{Reg}_M(L, 5) \geq 5$  hold.*

*Then  $\text{Base}_H(G) \leq 5$  and  $\text{Reg}_H(G, 5) \geq 5$ .*

Moreover, at the beginning of the proof of Theorem 2 from [6] the following statement is obtained.

**Lemma 3.** *If, for a group  $G$  and its subgroup  $H$ , the inequality  $\text{Base}_H(G) \leq 4$  holds, then  $\text{Reg}_H(G, 5) \geq 5$ .*

Thus by Theorems 1 and 2, Lemma 3, and [6, Theorem 2] we immediately obtain

**Theorem 4.** *Let  $H$  be a solvable  $\pi$ -Hall subgroup of  $G$ . Assume that each nonabelian composition factor of the socle of  $G/S(G)$ , where  $S(G)$  is the solvable radical of  $G$ , is isomorphic to either alternative, or sporadic, or exceptional group of Lie type. Then  $\text{Base}_H(G) \leq 5$ , i.e., there exist elements  $x, y, z, t$  of  $G$  such that the identity*

$$H \cap H^x \cap H^y \cap H^z \cap H^t = O_\pi(G)$$

*holds.*

## 1 Notations and preliminary results

Throughout by  $\pi$  a set of primes is denoted, while by  $\pi'$  we denote its complement in the set of all primes. A subgroup  $H$  of  $G$  is called a  $\pi$ -Hall subgroup, if the order  $|H|$  is divisible by primes from  $\pi$  only, while its index  $|G : H|$  is divisible by primes from  $\pi'$  only. The set of all  $\pi$ -Hall subgroups of  $G$  is denoted by  $\text{Hall}_\pi(G)$ . A subgroup  $H$  of  $G$  is called a Hall subgroup, if its order  $|H|$  and the index  $|G : H|$  are coprime. A group  $G$  is called *almost simple*, if there exists a nonabelian simple group  $S$  such that  $F^*(G) = S$ , where  $F^*(G)$  is the generalized Fitting subgroup of  $G$ . In other words,  $G$  is called almost simple, if there exists a simple group  $S$  such that  $S \simeq \text{Inn}(S) \leq G \leq \text{Aut}(S)$ .

**Lemma 5.** [7, Lemma 1] *Let  $G$  be a finite group and  $A$  be its normal subgroup. If  $H \in \text{Hall}_\pi(G)$ , then  $H \cap A \in \text{Hall}_\pi(A)$  and  $HA/A \in \text{Hall}_\pi(G/A)$ .*

**Lemma 6.** [8] *Let  $A$  be an abelian subgroup of a finite group  $G$ . Then there exists  $x \in G$  such that  $A \cap A^x \leq F(G)$ .*

Combining known results (see [9, Theorems 8.3–8.7]), we obtain the following

**Lemma 7.** *Let  $G$  be a simple group of Lie type over a field of characteristic  $p \in \pi$  and  $H$  be its solvable  $\pi$ -Hall subgroup. Then either  $H$  is included in a Borel subgroup of  $G$ , or one of the following holds:*

- (1)  $G = SL_3(2)$  or  $G = SL_3(3)$  and  $H$  is the stabilizer of a line or of a plain in the natural 3-dimensional module, i.e., there exist two classes of conjugate  $\pi$ -Hall subgroups in this case.
- (2)  $G = SL_4(2)$  or  $G = PSL_4(3)$  and  $H$  is the stabilizer of a two-dimensional subspace of the natural 4-dimensional module.
- (3)  $G = SL_5(2)$  or  $G = SL_5(3)$  and  $H$  is the stabilizer of a chain of subspaces  $V_0 < V_1 < V_2 < V_3 = V$  whose codimensions are in the set  $\{1, 2\}$  (i.e. two codimensions equal 2 and one codimension equals 1). There exist three classes of conjugate  $\pi$ -Hall subgroups in this case.

We recall some known technical results (see [10]). If  $G$  acts transitively on the set  $\Omega$ , then given  $x \in G$  by  $\text{fpr}(x)$  we denote the fixed point ratio of  $x$ , i.e.  $\text{fpr}(x) = |\text{fix}(x)|/|\Omega|$ , where  $\text{fix}(x) = \{\omega \in \Omega \mid \omega^x = \omega\}$ . If  $G$  acts transitively and  $H$  is a point stabilizer, then the following formulae is known

$$\text{fpr}(x) = \frac{|x^G \cap H|}{|x^G|}. \quad (1)$$

As it is noted in [11, Theorem 1.3], the base size can be bounded by using the following arguments. Assume that  $G$  acts faithfully and let  $Q(G, c)$  denote the probability that arbitrary chosen element of  $\Omega^c$  is not a  $G$ -regular point. Clearly,  $\text{Base}(G)$  is the minimal  $c$  such that  $Q(G, c) < 1$ . In particular, if  $Q(G, c) < 1$  then  $\text{Base}(G) \leq c$ . Clearly, an element of  $\Omega^c$  is not a  $G$ -regular point if and only if it is stable under the action of an element  $x$  of prime order. Notice also that the probability for arbitrary chosen element of  $\Omega^c$  to be stable under  $x$  is not greater than  $\text{fpr}(x)^c$ . Denote by  $\mathcal{P}$  the set of elements of  $G$  whose order is equal to a prime number. Let  $x_1, \dots, x_k$  be representatives of the conjugacy classes of elements from  $\mathcal{P}$ . Since  $G$  acts transitively, the formulae (1) shows that  $\text{fpr}(x)$  does not depend on the choice of the representative of a conjugacy class. Thus the following chain of inequalities holds.

$$Q(G, c) \leq \sum_{x \in \mathcal{P}} \text{fpr}(x)^c = \sum_{i=1}^k |x_i^G| \cdot \text{fpr}(x_i)^c =: \widehat{Q}(G, c). \quad (2)$$

In particular, we can use the upper bound for  $\text{fpr}(x)$  in order to bound  $\widehat{Q}(G, c)$  and so to bound  $Q(G, c)$ . The following lemma is the main technical tool for this bound.

**Lemma 8.** [10, Proposition 2.3] *Let  $G$  be a transitive group of permutations on  $\Omega$  and  $H$  be a point stabilizer. Assume that  $x_1, \dots, x_k$  are representatives of distinct conjugacy classes such that the inequalities  $\sum_i |x_i^G \cap H| \leq A$  and  $|x_i^G| \geq B$  hold for all  $i = 1, \dots, k$ . Then the inequality*

$$\sum_{i=1}^k |x_i^G| \cdot \text{fpr}(x_i)^c \leq B(A/B)^c$$

*holds for every  $c \in \mathbb{N}$ .*

Notice that for every subgroup  $H$  and every set  $x_1, \dots, x_k$  not containing the identity element the bound  $\sum_i |x_i^G \cap H| < |H|$  holds.

## 2 Technical results

Our notations for groups of Lie type agree with that of [12]. In particular, for every simple group of Lie type  $S$  over a field of characteristic  $p$  we fix a simple algebraic group  $\overline{G}$  of adjoint type and a Steinberg map  $\sigma$  so that  $S = O^{p'}(\overline{G}_\sigma)$ . Then  $\overline{G}_\sigma$  is the group of inner-diagonal automorphisms of  $S$  (we denote the group of inner-diagonal automorphisms of  $S$  by  $\widehat{S}$ ). We assume that a Borel  $\overline{B}$  and its maximal torus  $\overline{T}$  are chosen  $\sigma$ -invariant, and we denote  $\overline{B}_\sigma$  and  $\overline{T}_\sigma$  by  $B$  and  $T$  respectively. Recall that if  $S \in \{^2A_n(q), ^2D_n(q), ^2E_6(q)\}$ , then the definition field of  $S$  equals  $\mathbb{F}_{q^2}$ , if  $S = ^3D_4(q)$ , then the definition field of  $S$  equals  $\mathbb{F}_{q^3}$ , and the definition field of  $S$  equals  $\mathbb{F}_q$  in the remaining cases. For groups  $^2A_n(q), ^2D_n(q), ^2E_6(q)$  we also use the notations  $A_n^-(q), D_n^-(q), E_6^-(q)$  respectively. Notice also the known fact:  $Z(\overline{B}) \cap \overline{T} = Z(\overline{G})$  ( $= 1$ , if  $\overline{G}$  is of adjoint type) and  $Z(B) \cap T = Z(S)$  ( $= 1$ , if  $\overline{G}$  is of adjoint type).

**Lemma 9.** *Let  $G$  be a group of inner-diagonal automorphisms of a finite simple group of Lie type over a field of characteristic  $p$  (i.e.  $G = \overline{G}_\sigma$  for some connected simple algebraic group  $\overline{G}$  of adjoint type over an algebraically closed field of characteristic  $p$  and a Steinberg map  $\sigma$ ). Let  $B = U \rtimes T$  be a Borel subgroup of  $G$ , where  $U$  is a maximal unipotent subgroup of  $G$  and  $T$  is a Cartan subgroup of  $G$ . We denote the subgroup of monomial matrices containing  $T$  by  $N$  so that  $N/T \simeq W$  is the Weyl group of  $G$ . Let  $w_0 \in W$  be the unique element that maps all positive roots into negatives, and  $n_0$  be its preimage in  $N$ . Then there exists  $x \in U^{n_0}$  such that  $T^x \cap B = 1$ . In particular, there exist  $u, v \in O^{p'}(G)$  such that  $B \cap B^u \cap B^v = 1$ .*

*Proof.* Consider  $B^{n_0} = U^{n_0} \rtimes T$ . The Fitting subgroup  $F(U^{n_0} \rtimes T)$  equals  $U^{n_0}$  since  $Z(O^{p'}(G)) = 1$ . Otherwise, since  $U^{n_0}$  is a normal nilpotent subgroup of  $U^{n_0} \rtimes T$  we obtain that  $U^{n_0} \leq F(U^{n_0} \rtimes T)$ . If  $U^{n_0} \neq F(U^{n_0} \rtimes T)$ , then there exists  $1 \neq z \in T$  centralizing  $U^{n_0}$  and so lying in  $Z(O^{p'}(G)) = 1$ , a contradiction. Hence  $F(U^{n_0} \rtimes T) = U^{n_0}$  and by Lemma 6 there exists  $x \in U^{n_0}$  such that  $T \cap T^x = 1$ .

Notice that  $U^{n_0} \cap B = 1$ , so  $(U^{n_0} \rtimes T) \cap B = T$ . Since  $T^x \in U^{n_0} \rtimes T$  we obtain

$$1 = T^x \cap T = T^x \cap ((U^{n_0} \rtimes T) \cap B) = (T^x \cap (U^{n_0} \rtimes T)) \cap B = T^x \cap B,$$

whence the main statement of the lemma follows.

Now we prove “in particular”, i.e., we show that there exist  $u, v \in O^{p'}(G)$  such that  $B \cap B^u \cap B^v = 1$ . By construction,  $x \in U^{n_0} \leq O^{p'}(G)$  and  $1 = T^x \cap B = (B^{n_0} \cap B)^x \cap B = B \cap B^x \cap B^{n_0x}$ . The lemma is proven.  $\square$

Let  $S = O^{p'}(\overline{G}_\sigma)$  be a finite simple nontwisted group of Lie type over a field  $\mathbb{F}_q$  of characteristic  $p$ . A Cartan subgroup  $T \cap S$  of  $S$  can be obtained as  $\langle h_r(\lambda) \mid r \in \Pi, \lambda \in \mathbb{F}_q^* \rangle$  (see [12, Theorem 2.4.7]), where  $\Pi$  is a set of fundamental roots of the root system of  $S$ . Then a field automorphism  $\varphi$  of  $S$  can be chosen so that for every  $r \in \Pi, \lambda \in \mathbb{F}_q^*$  the identity  $h_r(\lambda)^\varphi = h_r(\lambda^p)$  holds. Moreover, a graph automorphism  $\tau$  corresponding to the symmetry  $\overline{\phantom{x}} : \Pi \rightarrow \Pi$  of the Dynkin diagram of  $S$  can be chosen so that for every  $r \in \Pi, \lambda \in \mathbb{F}_q^*$  the identity  $(h_r(\lambda))^\tau = h_{\bar{r}}(\bar{\lambda})$  holds, where  $\bar{\lambda} = \lambda$ , if all roots have the same length. Consider the subgroup  $A$  generated by so chosen field automorphism and graph automorphisms (there exist several graph automorphisms for the root system  $D_4$ ). It is well-known that  $\text{Aut}(S) = \widehat{S} \rtimes A$ . Moreover,  $A$  normalizes a Borel subgroup  $B$  containing the Cartan subgroup  $T$ . Since  $N_{\widehat{S}}(B) = B$  we obtain that  $N_{\text{Aut}(S)}(B) = B \rtimes A$ .

Now assume that  $S$  is a finite simple twisted group of Lie type distinct from a Suzuki group or a Ree group,  $L$  is a nontwisted group of Lie type and  $\psi$  is an automorphism

of  $L$  such that  $S = O^{p'}(L_\psi)$ . Let  $\bar{\cdot} : \Pi \rightarrow \Pi$  be the symmetry of the Dynkin diagram of a fundamental set of roots  $\Pi$  of the root system of  $L$  using for construction of  $\psi$ . Then a Cartan subgroup  $T \cap L$  of  $\widehat{L}$  can be written as  $\langle h_r(\lambda) \mid r \in \Pi, \lambda \in \mathbb{F}_q^* \rangle$ , and a field automorphism  $\varphi$  of  $S$  can be chosen so that for every  $r \in \Pi$ ,  $\lambda \in \mathbb{F}_q^*$  the equality  $(h_r(\lambda))^\varphi = h_{\bar{r}}(\lambda^p)$  holds. We set  $A = \langle \varphi \rangle$ , then  $\text{Aut}(S) = \widehat{S} \rtimes A$ , and there exists a Borel subgroup  $B$  of  $\widehat{S}$  such that the equality  $N_{\text{Aut}(S)}(B) = B \rtimes A$  holds.

**Lemma 10.** *In the introduced notations assume that, if  $S$  is not twisted, then the order  $q$  of the definition field  $\mathbb{F}_q$  of  $S$  is greater than 2. Moreover, if  $S = D_4(q)$ , assume also that  $q > 3$ . Assume also that  $S$  is neither a Suzuki group nor a Ree group. Then there exists  $x \in T \cap S$  such that  $C_A(x) = 1$ . In particular  $A \cap A^x = 1$ .*

*Proof.* If  $S$  is not twisted and is distinct from  $D_4(q)$ , then we can take  $x = h_r(\lambda)$ , where  $r \in \Pi$  is such that  $r \neq \bar{r}$  and  $\lambda$  is a generating element of the multiplicative group of  $\mathbb{F}_q$ . If  $S$  is twisted distinct from  ${}^3D_4(q)$ , then we can take  $x = h_r(\lambda)h_{\bar{r}}(\lambda^q)$ , where  $\lambda$  is a generating element of the multiplicative group of  $\mathbb{F}_{q^2}$  and  $r \neq \bar{r}$ . If  $S = {}^3D_4(q)$ , then we can take  $x = h_r(\lambda)h_{\bar{r}}(\lambda^q)h_{\bar{\bar{r}}}(\lambda^{q^2})$ , where  $\lambda$  is a generating element of the multiplicative group of  $\mathbb{F}_{q^3}$  and  $r \neq \bar{r}$ . Finally, if  $S = D_4(q)$  and  $q > 3$ , then there exist  $\lambda_1, \lambda_2 \in \mathbb{F}_q^* \setminus \{1\}$  such that  $\lambda_2 \notin \{\lambda_1^p, \lambda_1^{p^2}, \dots, \lambda_1^{q^2}\}$  and  $\lambda_1$  generates  $\mathbb{F}_q^*$ . Choose fundamental roots  $r, s$  so that there exists a nontrivial symmetry of the Dynkin diagram, permuting the roots. Then we can take  $x = h_r(\lambda_1)h_s(\lambda_2)$ .  $\square$

**Lemma 11.** *Let  $G$  be an almost simple group, whose simple socle  $S$  is a group of Lie type, satisfying the conditions of Lemma 10. Let  $B = U \rtimes T$  be a Borel subgroup of  $\widehat{S}$  and  $H = N_G(B)$ . Then there exist  $x, y, z \in S$ , such that  $H \cap H^x \cap H^y \cap H^z = 1$ .*

*Proof.* We use the notations introduced in Lemmas 9 and 10, in particular  $H \leq B \rtimes A$ . It is proven in Lemma 9 that there exists  $x \in U^{n_0} \leq S$  such that  $T^x \cap B = 1$ . In particular,  $B \cap B^{n_0} \cap B^{x^{-1}} = 1$ . Therefore  $H \cap H^{n_0} \cap H^{x^{-1}} \leq A$  and  $A \cap B = 1$ . By Lemma 10 there exists  $y \in T \cap S = (B \cap B^{n_0}) \cap S$  such that  $A \cap A^y = 1$ . Thus

$$(H \cap H^{n_0} \cap H^{x^{-1}}) \cap (H \cap H^{n_0} \cap H^{x^{-1}})^y = H \cap H^{n_0} \cap H^{x^{-1}} \cap H^{x^{-1}y} = 1,$$

whence the lemma follows.  $\square$

**Lemma 12.** *Let  $S$  be a simple exceptional group of Lie type over a field of characteristic  $p \notin \pi$  and  $H$  be a solvable  $\pi$ -Hall subgroup of  $S$ . Then one of the followings hold.*

- (1) *There exists a maximal torus  $T$  of  $S$  such that  $H \leq N(S, T)$  and  $|\pi(N(S, T)/T) \cap \pi| \leq 1$ .*
- (2)  *$S = {}^2G_2(3^{2n+1})$ ,  $\pi \cap \pi(S) = \{2, 7\}$ ,  $|S|_{\{2, 7\}} = 56$ ,  $H$  is a Frobenius group of order 56.*
- (3)  *$S \in \{G_2(q), F_4(q), E_6^{-\varepsilon}(q), {}^3D_4(q)\}$ , where  $\varepsilon \in \{+, -\}$  is chosen so that  $q \equiv \varepsilon 1 \pmod{4}$ ;  $2, 3 \in \pi$ ,  $\pi \cap \pi(S) \subseteq \pi(q - \varepsilon 1)$ ,  $H \leq N(S, T)$ , where  $T$  is a unique up to conjugation maximal torus such that  $N(S, T)$  contains a Sylow 2-subgroup of  $G$  and  $N(S, T)/T$  is a  $\{2, 3\}$ -group. Here  $N(S, T) := N_{\overline{G}}(\overline{T}) \cap S$ , where  $T = \overline{T} \cap S$  and  $S = O^{p'}(\overline{G}_\sigma)$ .*

*Proof.* If  $2 \notin \pi$  then by [13, Lemmas 7–14, Theorem 3] statement (1) of the Lemma holds.

If  $2 \in \pi$  and  $3 \notin \pi$ , then by [14, Lemma 5.1 and Theorem 5.2] (see also [9, Theorem 8.9]) either statement (1) or statement (2) of the lemma holds.

Finally, if  $2, 3 \in \pi$ , then  $S$  is neither a Suzuki group, nor a Ree group (since  $p \notin \pi$ ). By [15, Lemma 7.1–7.6] (see also [9, Theorem 8.15]) we have  $\pi \cap \pi(S) \subseteq \pi(q - \varepsilon 1)$ ,  $H \leq N(S, T)$ , where  $T$  is a unique up to conjugation maximal torus such that  $N(S, T)$  contains a Sylow 2-subgroup of  $S$  and either  $N(S, T)/T$  is a  $\{2, 3\}$ -group, or  $N(S, T)/T$  is a Weyl group of the root system of  $S$ . Since for root systems  $E_6, E_7, E_8, F_4, G_2$  the Weyl groups are either  $\{2, 3\}$ -groups, or unsolvable, we obtain that if  $S \in \{E_6^\varepsilon(q), E_7(q), E_8(q)\}$ , then  $H$  is unsolvable, whence statement (3) of the lemma.  $\square$

**Corollary 13.** *Let  $S$  be a simple exceptional group of Lie type over a field of characteristic  $p \notin \pi$ ,  $S$  is neither a Suzuki group, nor a Ree group, and  $H$  is a solvable  $\pi$ -Hall subgroup of  $S$ . Then the following statements hold.*

- (1) *If  $S = E_8(q)$ , then  $|H| \leq (q + 1)^8 \cdot 2^{14}$ .*
- (2) *If  $S = E_7(q)$ , then  $|H| \leq (q + 1)^7 \cdot 2^{10}$ .*
- (3) *If  $S = E_6^\varepsilon(q)$ , then  $|H| \leq (q + 1)^6 \cdot 2^7$ .*
- (4) *If  $S = F_4(q)$ , then  $|H| \leq (q + 1)^4 \cdot 2^7 \cdot 3^2$ .*
- (5) *If  $S = G_2(q)$ , then  $|H| \leq (q + 1)^2 \cdot 12$ .*
- (6) *If  $S = {}^3D_4(q)$ , then  $|H| \leq \max\{(q^2 + q + 1)^2, (q + 1)^2 \cdot 48\}$ .*

### 3 Proof of the Main Theorem.

We proceed by considering distinct possible cases for the simple socle  $S$  of  $G$  and the structure of its  $\pi$ -Hall subgroup  $H$ . If  $S$  is either a Suzuki group or a Ree group, then by [10, Tables 3 and 4] it follows that for every subgroup  $H$  of  $G$  the inequality  $\text{Base}_H(G) \leq 3$  holds. So we assume later that  $S$  is neither a Suzuki group, nor a Ree group.

#### 3.1 $S$ is a simple group of Lie type over a field of characteristic $p \in \pi$ .

By Lemma 5,  $H \cap \widehat{S} \in \text{Hall}_\pi(\widehat{S})$ , so in this case for  $H \cap \widehat{S}$  Lemma 7 holds. Assume first that  $H \cap \widehat{S}$  lies in a Borel subgroup of  $\widehat{S}$ . If  $S$  is a nontwisted group of Lie type over a field of order two, then  $H$  is a 2-group. By [2] the inequality  $\text{Base}_H(G) \leq 3$  holds. If  $S = D_4(3)$ , then  $H$  is a 3-group. By [2] the inequality  $\text{Base}_H(G) \leq 3$  holds. Assume that  $S$  is not a nontwisted group of Lie type over a field of two elements, and  $S \not\cong D_4(3)$ . Then  $H \leq N_G(U) = N_G(B)$  and by Lemma 11 the inequality  $\text{Base}_H(G) \leq 4$  holds. If one of statements (1)–(3) of Lemma 7 is satisfied, then  $S$  is a classical group and calculations by using [16] show that in any case  $\text{Base}_H(G) \leq 5$  and  $\text{Reg}_H(G, 5) \geq 5$ .

#### 3.2 $S$ is a simple exceptional group of Lie type over a field of characteristic $p \notin \pi$ .

Assume that  $S = E_8(q)$ . We use Lemma 8. If  $x$  is a unipotent element, then  $x^G \cap H = \emptyset$ . If  $x$  is a semisimple element from  $G = \widehat{G}$ , then by [17, Table 2] it follows that the maximum of orders of centralizers of semisimple elements in  $E_8(q)$  is not greater than

$$q^{64}(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^{10} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)^2,$$

whence  $|x^G| > q^{112}$ . Clearly, the inequality  $|x^G| > q^{112}$  holds in case, when  $x$  is a field automorphism. So for  $c = 2$  we obtain

$$\widehat{Q}(G, 2) \leq ((q+1)^8 \cdot 2^{14})^2 / (q^{112}) < 1$$

for every  $q \geq 2$ . Hence,  $\text{Base}_H(G) \leq 2$ .

Assume that  $G = E_7(q)$ . We again use Lemma 8. If  $x$  is a unipotent element, then  $x^G \cap H = \emptyset$ . If  $x$  is a semisimple element from  $\widehat{G}$ , then by [17, Table 1] it follows that the maximum of orders of centralizers of semisimple elements in  $E_7(q)$  is not greater than

$$q^{31}(q^2 - 1)^2(q^4 - 1)(q^6 - 1)^2(q^8 - 1)(q^{10} - 1),$$

whence  $|x^G| > (1/2)q^{64}$ . Clearly, the inequality  $|x^G| > (1/2)q^{64}$  holds in case, when  $x$  is a field automorphism. So for  $c = 2$  we obtain

$$\widehat{Q}(G, 2) \leq ((q+1)^7 \cdot 2^{20})^2 \cdot 2 / (q^{64}) < 1$$

for every  $q \geq 2$ . Hence  $\text{Base}_H(G) \leq 2$ .

Assume that  $G = E_6^\epsilon(q)$ . As above, we obtain that  $x$  is either a semisimple element from  $\widehat{G}$ , or does not lie in  $\widehat{G}$ . If  $x$  is a semisimple element, then by [18, Table 1 and Case  $E_6(q)$ ] it follows that the maximum of orders of centralizers of semisimple elements in  $E_6^\epsilon(q)$  is not greater than

$$q^{20}(q - \epsilon 1)(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1)(q^5 - \epsilon 1),$$

whence  $|x^G| > \frac{1}{3}q^{30}$ . Clearly, the inequality  $|x^G| > \frac{1}{3}q^{30}$  holds in case, when  $x$  is either a field, or a graph-field automorphism. If  $x$  is a graph automorphism, then

$$|x^G| = |E_6^\epsilon| / |F_4(q)| \geq \frac{1}{3}q^{12}(q^5 - 1)(q^9 - 1).$$

So for  $c = 4$  we obtain

$$\widehat{Q}(G, 2) \leq \frac{(q+1)^{24} \cdot 2^{28} \cdot 3^3}{q^{36} \cdot (q^5 - 1)^3 \cdot (q^9 - 1)^3} < 1$$

for every  $q \geq 2$ . Hence  $\text{Base}_H(G) \leq 4$ .

Assume that  $G = F_4(q)$ . Again we may assume that  $x$  either is a semisimple element from  $G = \widehat{G}$  or does not lie in  $\widehat{G}$ . If  $x$  is a semisimple element, then by [18, Table 2] it follows that the maximum of orders of centralizers of semisimple elements in  $F_4(q)$  is not greater than

$$q^{16}(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1),$$

whence  $|x^G| > q^{16}$ . Clearly, the inequality  $|x^G| > q^{16}$  holds for every  $x$  not lying in  $G$ . So for  $c = 4$  we obtain

$$\widehat{Q}(G, 2) \leq \frac{(q+1)^{16} \cdot 2^{28} \cdot 3^8}{q^{48}} < 1$$

for every  $q \geq 3$ . So for  $q \geq 3$  the inequality  $\text{Base}_H(G) \leq 4$  holds. If  $q = 2$ , then in view of the condition  $p \notin \pi$  we obtain that the order  $|H|$  is odd. By [13, Lemma 8] we obtain that either  $H$  is a Sylow 3-subgroup of  $G$ , or  $H$  is abelian. Hence the inequality  $\text{Base}_H(G) \leq 3$  holds: in the first case by [2], and in the second case by Lemma 6.

Assume that  $G = G_2(q)$ . As above we may assume that  $x$  either is a semisimple element from  $G = \widehat{G}$ , or does not lie in  $\widehat{G}$ . If  $x$  is semisimple, then by [18, Table 4] it

follows that the maximum of orders of centralizers of semisimple elements in  $F_4(q)$  is not greater than

$$q^2(q^2 - 1)(q^3 + 1),$$

whence  $|x^G| \geq q^4(q^3 - 1)$ . Clearly, the inequality  $|x^G| \geq q^4(q^3 - 1)$  holds for every  $x$  not lying in  $G$ . So for  $c = 4$  we obtain

$$\widehat{Q}(G, 2) \leq \frac{(q+1)^8 \cdot 12^4}{q^{12} \cdot (q^3 - 1)^3} < 1$$

for every  $q \geq 3$ . Hence for  $q \geq 3$  the inequality  $\text{Base}_H(G) \leq 4$  holds. If  $q = 2$ , then by the condition  $p \notin \pi$  we obtain that the order  $|H|$  is odd. By [13, Lemma 7] we obtain that either  $H$  is a Sylow 3-subgroup of  $G$ , or  $H$  is abelian. So the inequality  $\text{Base}_H(G) \leq 3$  holds: in the first case by [2], and in the second case by Lemma 6.

If  $G = {}^3D_4(q)$ , then by [18, Table 7] it is easy to get the bound  $|x^G| > q^{16}$ . Using the bound we obtain that for  $q \geq 2$  the inequality  $\text{Base}_H(G) \leq 4$  holds. The Main Theorem is proven.

Notice that for the case  $p \in \pi$  we also prove the following

**Theorem 14.** *Let  $G$  be a finite almost simple group, whose simple socle is isomorphic to a group of Lie type over a field of characteristic  $p \in \pi$ . Assume that  $H$  is a solvable  $\pi$ -Hall subgroup of  $G$ . Then the inequalities  $\text{Base}_H(G) \leq 5$  and  $\text{Reg}_H(G, 5) \geq 5$  hold.*

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